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Securely Implementable Social Choice Functions in Divisible and Non-Excludable Public Good Economies with Quasi-Linear Utility Functions

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Securely Implementable Social Choice Functions in Divisible and Non-Excludable Public Good Economies with Quasi-Linear Utility Functions *

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Abstract

This paper studies the possibility of secure implementation (Saijo, T., T. Sjöström, and T. Yamato (2007) "Secure Implementation," *Theoretical Economics* 2, pp.203-229) in divisible and non-excludable public good economies with quasi-linear utility functions. In this economies, although Saijo, Sjöström, and Yamato (2007) showed that the Groves mechanisms (Groves, T. (1973) "Incentives in Teams," *Econometrica* 41, pp.617-631) are securely implementable when the valuation functions of the public good are (a) differentiable and concave and (b) fixed and the agents are identified with their parameters respectively, this paper presents the following negative result: securely implementable social choice functions are dictatorial or constant when the valuation functions of the public good are strictly increasing and strictly concave.

Keywords: Secure Implementation, Dominant Strategy Implementation, Nash Implementation, Strategy-Proofness, Non-Excludable Public Good.

JEL Classification: C72, D61, D63, D71, H41.

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1 Introduction

This paper considers divisible and non-excludable public good economies in which $n \geq 2$ agents collectively decide (a) how much of the public good (e.g., seawalls, protection forests, and storm sewers) should be provided and (b) how the cost should be shared among the agents. [1] These decisions are made to achieve a goal characterized by a **social choice function** which associates an outcome with the agents' preferences. This paper studies strategy-proof social choice functions in such public good economies with quasi-linear utility functions. **Strategy-proofness** requires that truthful revelation is a weakly dominant strategy for the agent.

Although strategy-proofness is a desirable property, some experimental studies have questioned the performance of strategy-proof mechanisms. [2] On the basis of these observations, Saijo, Sjöström, and Yamato (2007) introduced **secure implementation** which is defined as double implementation in dominant strategy equilibria and Nash equilibria. [3] They showed that the social choice function is **securely implementable** if and only if it satisfies strategy-proofness and the rectangular property (Saijo, Sjöström, and Yamato, 2007). In addition, they showed that the rectangular property is in general equivalent to the combination of **strong non-bossiness** (Ritz, 1983) and the **outcome rectangular property** (Saijo, Sjöström, and Yamato, 2007). Strong non-bossiness requires that the agent cannot change the allocation by changing the agent's revelation while maintaining the agent's utility. The outcome rectangular property requires that the allocation does not change by changing all the agents' revelations, each of whom does not change the allocation. On the basis of these characterizations, some researchers have studied the possibility of secure implementation in several environments and illustrated the difficulty of finding securely implementable social choice functions with desirable properties. [4]

In divisible and non-excludable public good economies with quasi-linear utility functions, which are also considered in this paper, Saijo, Sjöström, and Yamato (2007) showed that the Groves mechanisms (Groves, 1973) are securely implementable when the valuation functions of the public good are (a) differentiable and concave and (b) fixed and the agents are identified with their parameters respectively. Because this assumption of the valuation functions is so restrictive, this paper considers the following more reasonable assumption: the valuation functions of the public good are strictly increasing and strictly concave. Under this assumption of the valuation functions, this paper demonstrates that securely implementable social choice functions are dictatorial or constant in divisible and non-excludable public good economies with quasi-linear utility functions.

The main result presented here depends on the results of Barberà and Peleg (1990). In a voting environment in which the set of alternatives is a metric space and each agent's preference is represented by a continuous utility function, they showed that if the social choice function satisfies strategy-proofness and it's range contains at least three alternatives, then it is dictatorial. Because secure implementability

[1] See Clarke (1971), Moulin (1994), and Serizawa (1996, 1999) for non-excludable public good economies.

[2] See Chen (2008) for a survey of experimental studies on strategy-proof mechanisms in non-excludable public good economies.

[3] Cason, Saijo, Sjöström, and Yamato (2006) conducted experiments on secure implementation and suggested that it might be a benchmark for constructing a mechanism which works well in practice.

[4] See Mizukami and Wakayama (2005), Saijo, Sjöström, and Yamato (2003, 2007), Fujinaka and Wakayama (2008, 2011), Berga and Moreno (2009), Bochet and Sakai (2010), Kumar (2013), and Nishizaki (2011, 2012, 2013, 2014) for theoretical results on secure implementation.

reduces the problem of providing a divisible and non-excludable public good with the cost shares to such a voting environment in the model presented here, this paper presents the main result on the basis of the results of Barberà and Peleg (1990).

The remainder of this paper is organized as four sections. Section 2 introduces the model presented here and Section 3 the main properties of social choice functions. Section 4 presents the main result and the relationship between it and some previous studies. Section 5 concludes this paper. Some preliminary results and the proof of the main result of this paper are presented in Appendix.

2 Model

Let $I \equiv \{1, \ldots, n\}$ be the set of **agents**, where $n \geq 2$. Let $Y \subseteq \mathbb{R}_+ \equiv \{r \in \mathbb{R} | r \geq 0\}$ be a convex set of **production levels of the public good** and $c\colon Y \to \mathbb{R}_+$ be the **cost function**. In the model presented here, a production level of the public good is equal to consumption of the public good for each agent. For each $i \in I$, let $(y, x_i) \in Y \times \mathbb{R}_+$ be a **consumption bundle for agent *i***, where $x_i \in \mathbb{R}_+$ is a **cost share of the public good for agent *i***. Let (y, x) be an **allocation**, where $x \equiv (x_i)_{i \in I}$ is a profile of cost shares of the public good, and $Z \equiv \{(y, x) \in Y \times \mathbb{R}^n_+ | c(y) \leq \sum_{i \in I} x_i\}$ be the set of **feasible allocations**.

This paper assumes that each agent's preference is represented by a quasi-linear utility function. For each $i \in I$, let $u_i\colon Y \times \mathbb{R}_+ \to \mathbb{R}$ be an **utility function for agent *i*** such that there is $v_i\colon Y \to \mathbb{R}$, called a **valuation function of the public good for agent *i***, and for each $(y, x_i) \in Y \times \mathbb{R}_+$, $u_i(y, x_i) = v_i(y) - x_i$. This paper also assumes that each valuation function of the public good is strictly increasing and strictly concave. For each $i \in I$, let V_i be the set of all valuation functions of the public good for agent i with such characteristics. Let $v \equiv (v_k)_{k \in I}$ be a profile of valuation functions of the public good and $V \equiv \prod_{k \in I} V_k$ be the set of the profiles. For each $i \in I$, let $v_{-i} \equiv (v_k)_{k \in I \setminus \{i\}}$ be a profile of valuation functions of the public good other than agent i and $V_{-i} \equiv \prod_{k \in I \setminus \{i\}} V_k$ be the set of the profiles. For each $i, j \in I$, let $v_{-i,j} \equiv (v_k)_{k \in I \setminus \{i,j\}}$ be a profile of valuation functions of the public good other than agents i and j. For each $S, S', S'' \subseteq I$, where these sets are mutually disjoint and $S \cup S' \cup S'' = I$, and each $v, v', v'' \in V$, let $(v_S, v'_{S'}, v''_{S''})$ be the profile of valuation functions of the public good, where agent $i \in S$ has v_i, agent $i \in S'$ has v'_i, and agent $i \in S''$ has v''_i.

Let $f\colon V \to Z$ be a **social choice function**. For each $v \in V$, let $(y(v), x(v)) \in Z$ be the allocation under the social choice function f at the profile of valuation functions of the public good v and $(y(v), x_i(v))$ be the consumption bundle for agent $i \in I$ at the allocation $(y(v), x(v))$.

3 Properties of Social Choice Functions

Saijo, Sjöström, and Yamato (2007, Theorem 1) characterized securely implementable social choice functions by **strategy-proofness** and the rectangular property (Saijo, Sjöström, and Yamato, 2007). Strategy-proofness requires that truthful revelation is a weakly dominant strategy for the agent.

Definition 1. The social choice function f satisfies **strategy-proofness** if and only if for each $v, v' \in V$ and each $i \in I$, $v_i(y(v_i, v'_{-i})) - x_i(v_i, v'_{-i}) \geq v_i(y(v'_i, v'_{-i})) - x_i(v'_i, v'_{-i})$.

In addition, Saijo, Sjöström, and Yamato (2007, Proposition 3) showed that the rectangular property is in general equivalent to the combination of **strong non-bossiness** (Ritz, 1983) and the **outcome rect-**

angular property (Saijo, Sjöström, and Yamato, 2007). Strong non-bossiness requires that if the agent does not change the agent's "utility" by changing the agent's revelation, then the allocation also does not change by the change of the revelation. The outcome rectangular property requires that if each agent cannot change the "allocation" by changing the agent's revelation, then the allocation does not change by changing all the agents' revelations.

Definition 2. The social choice function f satisfies **strong non-bossiness** if and only if for each $v, v' \in V$ and each $i \in I$, if $v_i(y(v_i, v'_{-i})) - x_i(v_i, v'_{-i}) = v_i(y(v'_i, v'_{-i})) - x_i(v'_i, v'_{-i})$, then $(y(v_i, v'_{-i}), x(v_i, v'_{-i})) = (y(v'_i, v'_{-i}), x(v'_i, v'_{-i}))$.

Definition 3. The social choice function f satisfies the **outcome rectangular property** if and only if for each $v, v' \in V$, if $(y(v_i, v'_{-i}), x(v_i, v'_{-i})) = (y(v'_i, v'_{-i}), x(v'_i, v'_{-i}))$ for each $i \in I$, then $(y(v), x(v)) = (y(v'), x(v'))$.

Theorem 1 (Saijo, Sjöström, and Yamato, 2007)**.** *The social choice function is **securely implementable** if and only if it satisfies **strategy-proofness**, **strong non-bossiness**, and the **outcome rectangular property**.*

This paper shows that securely implementable social choice functions are dictatorial or constant in divisible and non-excludable public good economies with quasi-linear utility functions whose valuation functions of the public good are strictly increasing and strictly concave.

Definition 4. The social choice function f is **dictatorial** if and only if there is $i \in I$ such that for each $v, v' \in V$, $v_i(y(v)) - x_i(v) \geq v_i(y(v')) - x_i(v')$. [5]

Definition 5. The social choice function f is **constant** if and only if for each $v, v' \in V$, $(y(v), x(v)) = (y(v'), x(v'))$.

4 Main Result

Saijo, Sjöström, and Yamato (2007, Lemma 3) showed that the Groves mechanisms are securely implementable in some of divisible and non-excludable public good economies with quasi-linear utility functions. A major difference between the model presented here and those of Saijo, Sjöström, and Yamato (2007) is the assumption of valuation functions of the public good. Although this paper assumes that the valuation functions of the public good are strictly increasing and strictly concave, Saijo, Sjöström, and Yamato (2007) assumed that they are (a) differentiable and concave and (b) fixed and the agents are identified with their parameters respectively (e.g., for each $i \in I$ and each $y \in Y$, $v_i(y) = \theta_i b(y)$, where $\theta_i \in \mathbb{R}$ and $b\colon Y \to \mathbb{R}$ is differentiable and concave). [6] Under this assumption of Saijo, Sjöström, and Yamato (2007), the Groves mechanisms satisfy strong non-bossiness and the outcome rectangular property in addition to strategy-proofness. In the model presented here, the Groves mechanisms do not satisfy both properties. The following example shows a social choice function satisfying strategy-proofness and the outcome rectangular property, but not strong non-bossiness.

[5] Note that this dictatorship is required on the range of the social choice function, but not on the set of all feasible allocations.

[6] Because this assumption does not require the strict increasingness, single-peaked valuation functions of the public good are also considered by Saijo, Sjöström, and Yamato (2007). In addition, they assumed that the cost function is differentiable and convex, the set of maximizers of the sum of all the agents' benefits from the consumption is singleton, and the element is in the interior of the set of production levels of the public good.

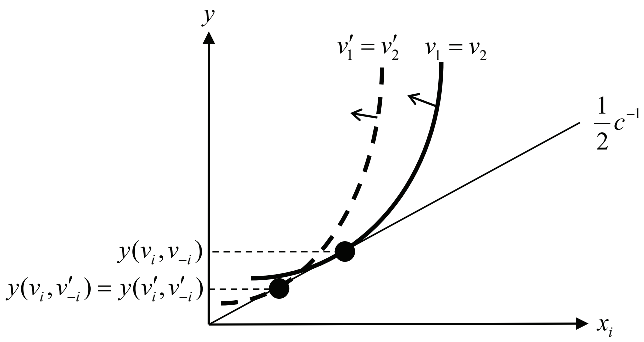

Figure 1: A violation of the outcome rectangular property under the conservative equal cost sharing mechanism, where $n = 2$ and c is a liner cost function

Example 1. Let f be the following social choice function: there is $y \in Y$ such that for each $v \in V$, $y(v) = y$ and $x_i(v) = -\{\sum_{k \in I \setminus \{i\}} v_k(y(v)) - c(y(v))\}$ for each $i \in I$. We find that f satisfies strategy-proofness and the outcome rectangular property. In addition, we find that f does not satisfy strong non-bossiness because the agent can change other agents' cost shares of the public good by changing the agent's revelation while maintaining the agent's utility.

Although the cost shares of the public good under the social choice function in Example 1 are contained in those of the Groves mechanisms, the consumption of the public good does not maximize the sum of all the agents' benefits from the consumption. Even if the consumption maximizes it, then the social choice function does not satisfy strong non-bossiness. Together with Theorem 1, this implies that the Groves mechanisms are not securely implementable in the model presented here.

The conservative equal cost sharing mechanism (Moulin, 1994) is a well-known cost sharing scheme satisfying strategy-proofness and non-bossiness (Satterthwaite and Sonnenschein, 1981) in non-excludable public good economies. [7] For each $i \in I$, let $t_i \colon Y \to \mathbb{R}_+$ be a **cost sharing function for agent *i***. The social choice function f is a **cost sharing scheme** if and only if there are cost sharing functions $t_1, \cdots, t_n$ such that for each $v \in V$ and each $i \in I$, $x_i(v) = t_i(y(v))$. Given a cost sharing scheme, for each $v \in V$ and each $i \in I$, let $B_i(v_i, t_i, y(V)) \equiv \{y \in y(V) | v_i(y) - t_i(y) \geq v_i(y') - t_i(y') \text{ for each } y' \in y(V)\}$ be the set of utility maximizers for agent i in the range of consumption of the public good $y(V)$ at the profile of valuation functions of the public good v and $b_i(v_i, t_i, y(V)) \equiv \max B_i(v_i, t_i, y(V))$. The social choice function f is the **conservative equal cost sharing mechanism** if and only if f is a cost cost sharing scheme such that for each $v \in V$, $t_i(y(v)) = c(y(v))/n$ for each $i \in I$, where $y(v) = \min\{b_i(v_i, t_i, y(V))\}_{i \in I}$. In the model presented here, if the cost function c is convex, then the conservative equal cost sharing mechanism satisfies strong non-bossiness, but not the outcome rectangular property (see Figure 1). [8] Together with Theorem 1, this implies that the conservative equal cost sharing mechanism is not securely implementable.

It is well-known that a social choice function satisfying strategy-proofness and non-bossiness is a

[7] The social choice function f satisfies **non-bossiness** if and only if for each $v, v' \in V$ and each $i \in I$, if $(y(v_i, v'_{-i}), x_i(v_i, v'_{-i})) = (y(v'_i, v'_{-i}), x_i(v'_i, v'_{-i}))$, then $(y(v_i, v'_{-i}), x(v_i, v'_{-i})) = (y(v'_i, v'_{-i}), x(v'_i, v'_{-i}))$.

[8] In this figure, $((y(v), x(v)))$ is the allocation induced by the unique weakly dominant strategy equilibrium and $((y(v'), x(v')))$ is an allocation induced by a "bad" Nash equilibrium.

cost sharing scheme (Lemma 5 in Appendix). Because strong non-bossiness is in general stronger than non-bossiness, a social choice function satisfying strategy-proofness and strong non-bossiness is also a cost sharing scheme. In addition, secure implementability implies that the range of consumption of the public good is closed (Proposition 3 in Appendix) and convex (Proposition 4 in Appendix). These imply that the problem of providing a divisible and non-excludable public good with the cost shares is reduced to a voting environment in which the set of alternatives is equivalent to the range of consumption of the public good, which is a closed interval by secure implementability. In such a voting environment, Barberà and Peleg (1990, Theorem 3.1) showed the following negative result.

Theorem 2 (Barberà and Peleg, 1990)**.** *In a voting environment in which the set of alternatives is a metric space and each agent's preference is represented by a continuous utility function defined over the set, if the social choice function satisfies* ***strategy-proofness*** *and it's range contains at least three alternatives, then it is* ***dictatorial****.*

Note that Theorem 2 implies that any strategy-proof social choice function is constant if it is not dictatorial and it's range is a closed interval in a voting environment considered by Barberà and Peleg (1990).

In the model presented here, each utility function induces a continuous preference defined over the range of consumption of the public good associated with secure implementability (Remark 2 in Appendix). [9] On the basis of Theorem 2, the following main result is presented.

Theorem 3. *If the social choice function f satisfies* ***strategy-proofness****,* ***strong non-bossiness****, and the* ***outcome rectangular property****, then f is* ***dictatorial*** *or* ***constant****.* [10]

Theorem 3 is tight. The argument about the conservative equal cost sharing mechanism showed the necessity of the outcome rectangular property and Example 1 the necessity of strong non-bossiness. In addtion, the following example shows the necessity of strategy-proofness.

Example 2. Let f be the following social choice function: there is $y \in Y$ such that for each $v \in V$, $y(v) = y$ and $x_i(v) = -\{\sum_{k \in I} v_k(y(v)) - c(y(v))\}$ for each $i \in I$. We find that f satisfies strong non-bossiness and the outcome rectangular property. In addition, we find that f does not satisfy strategy-proofness because the agent benefits from untruthful revelation that changes the agent's cost share of the public good in the agent's favor.

Together with Theorem 1, Theorem 3 implies the following negative result on secure implementation in divisible and non-excludable public good economies with quasi-linear utility functions whose valuation functions of the public good are strictly increasing and strictly concave.

Corollary. *If the social choice function is* ***securely implementable****, then it is* ***dictatorial*** *or* ***constant****.*

5 Conclusion

This paper studies the possibility of secure implementation in divisible and non-excludable public good economies with quasi-linear utility functions. In this economies, although Saijo, Sjöström, and Yamato

[9] It is unclear whether the preferences are also single-peaked. Together with the result of Saijo, Sjöström, and Yamato (2007, Theorem 8), we have the same result as Theorem 3 even if the preferences are single-peaked.

[10] See Appendix for the formal proof of this theorem.

(2007) showed that the Groves mechanisms are securely implementable when the valuation functions of the public good are (a) differentiable and concave and (b) fixed and the agents are identified with their parameters respectively, the results presented here implied that securely implementable social choice functions are dictatorial or constant when the valuation functions of the public good are strictly increasing and strictly concave. This result depends on the results of Barberà and Peleg (1990) because secure implementability reduces the problem of providing a divisible and non-excludable public good with the cost shares to a voting environment considered by Barberà and Peleg (1990).

Investigating securely implementable social choice functions in divisible and "excludable" public good economies is an interesting research topic because there are non-trivial ones (e.g. a convex cost sharing mechanism under which the convexity of the cost sharing functions is established on the range of consumption of the public good and each agent is assigned the consumption bundle which maximizes the agent's utility according to the agent's cost sharing function) although the serial cost sharing mechanism (Molin, 1994) is not securely implementable. [11] In addition, Saijo, Sjöström, and Yamato (2007) and Kumar (2013) showed a positive result on secure implementation in the problems of providing a divisible private good with monetary transfers and Nishizaki (2014) in pure exchange economies with Leontief utility functions. These environments suggest further research on secure implementation.

Appendix: Some Preliminary Results and Proof of Theorem 3

This paper demonstrates some preliminary results on strategy-proofness, strong non-bossiness, and the outcome rectangular property. These results specify the characteristics of the option sets, the cost shares of the public good, and the range of consumption of the public good under a securely implementable social choice function.

For each $i \in I$ and each $v'_{-i} \in V_{-i}$, let $O_i(v'_{-i}) \equiv \{y \in Y | \text{there is } v_i \in V_i \text{ such that } y(v_i, v'_{-i}) = y\}$ be the **option set for agent i at v_{-i} under the social choice function f**, that is, the set of consumption of the public good, which the agent can induce given f and v_{-i} and $O_i(V_{-i}) \equiv \cup_{v'_{-i} \in V_{-i}} O_i(v'_{-i})$. In addition, let $y(V) \equiv \{y \in Y | \text{there is } v \in V \text{ such that } y(v) = y\}$ be the **range of consumption of the public good under the social choice function f**, that is, the set of consumption of the public good, which all the agents can induce given f. By definition, $y(V) \supseteq O_i(V_{-i})$ for each $i \in I$. Lemma 1 shows that both sets are equivalent. [12]

Lemma 1. *For each $i \in I$, $y(V) = O_i(V_{-i})$.*

The cost sharing scheme f is (a) **strictly increasing** if and only if for each $i \in I$, each $v'_{-i} \in V_{-i}$, and each $y, y' \in O_i(v'_{-i})$, where $y < y'$, $t_i(y) < t_i(y')$, (b) **lower semi-continuous** if and only if for each $i \in I$, each $v'_{-i} \in V_{-i}$, each $y \in O_i(v'_{-i})$, and each $\varepsilon > 0$, there is a neighborhood $U \subseteq O_i(v'_{-i})$ of y such that $t_i(y') \geq t_i(y) - \varepsilon$ for each $y' \in U$, (c) **upper semi-continuous** if and only if for each $i \in I$ and each $v'_{-i} \in V_{-i}$, each $y \in O_i(v'_{-i})$, and each $\varepsilon > 0$, there is a neighborhood $U \subseteq O_i(v'_{-i})$ of y such that $t_i(y') \leq t_i(y) + \varepsilon$ for each $y' \in U$, and (d) **continuous** if and only if for each $i \in I$ and each $v'_{-i} \in V_{-i}$, t_i is upper semi-continuous and lower semi-continuous on $O_i(v'_{-i})$. [13]

[11] Secure implementability of the serial cost sharing mechanism was noted by Yuji Fujinaka.

[12] See Nishizaki (2016) for a proof of this lemma.

[13] Note that these properties are required on the option sets, but not on the set of consumption of the public good.

A1. Results on Strategy-Proofness

Lemma 2 shows the strict increasingness of cost sharing schemes satisfying strategy-proofness. [14] On the other hand, Lemma 3 shows the lower semi-continuity of cost sharing schemes satisfying strategy-proofness on the basis of the continuity of valuation functions of the public good.

Lemma 2. *If the cost sharing scheme satisfies **strategy-proofness**, then it is **strictly increasing**.*

Lemma 3. *If the cost sharing scheme f satisfies **strategy-proofness**, then it is **lower semi-continuous**.*

Proof. To the contrary, we suppose that f is not lower semi-continuous. This implies that there are $i \in I$ and $v'_{-i} \in V_{-i}$ such that t_i is not lower semi-continuous on $O_i(v'_{-i})$. In addition, there is $v_i \in V_i$ such that t_i is not lower semi-continuous at $y(v_i, v'_{-i})$. This implies that there is $\varepsilon \in \mathbb{R}_+$ such that for each neighborhood $U \subseteq O_i(v'_{-i})$ of $y(v_i, v'_{-i})$,

$$t_i(y') < t_i(y(v_i, v'_{-i})) - \varepsilon \tag{1}$$

for some $y' \in U$. By the continuity of valuation functions of the public good, we can take the neighborhood to satisfy the following condition:

$$v_i(y(v_i, v'_{-i})) - v_i(y') < \varepsilon. \tag{2}$$

Because $U \subseteq O_i(v'_{-i})$, there is $v'_i \in V_i$ such that $y(v'_i, v'_{-i}) = y'$ and we find that $v_i(y(v_i, v'_{-i})) - v_i(y(v'_i, v'_{-i})) < \varepsilon < t_i(y(v_i, v'_{-i})) - t_i(y(v'_i, v'_{-i}))$ by (1) and (2). This implies that $v_i(y(v_i, v'_{-i})) - x_i(v_i, v'_{-i}) < v_i(y(v'_i, v'_{-i})) - x_i(v'_i, v'_{-i})$ and contradicts strategy-proofness. □

Proposition 1 shows the closedness of the option sets under a cost sharing scheme satisfying strategy-proofness on the basis of Lemma 2 and the continuity and strict increasingness of valuation functions of the public good.

Proposition 1. *Suppose that the cost sharing scheme f satisfies **strategy-proofness**. For each $i \in I$ and each $v'_{-i} \in V_{-i}$, $O_i(v'_{-i})$ is **closed**.*

Proof. To the contrary, we suppose that there are $i \in I$ and $v'_{-i} \in V_{-i}$ such that $O_i(v'_{-i})$ is not closed. This implies that we can take $y \in \bar{O}(v'_{-i}) \setminus O_i(v'_{-i})$, where $\bar{O}(v'_{-i})$ is the closure of $O_i(v'_{-i})$. We have the following three situations according to the relationship between y and $O_i(v'_{-i})$.

Situation 1. $y = \inf O_i(v'_{-i})$

Let $x_i^H \equiv \inf\{x_i \in \mathbb{R}_+ | \text{there is } v_i \in V_i \text{ such that } t_i(y(v_i, v'_{-i})) = x_i\}$. By Lemma 2, the definition of x_i^H, and the continuity and strict increasingness of valuation functions of the public good, we can take $v_i \in V_i$ such that $v_i(y) - v_i(y(v''_i, v'_{-i})) > x_i^H - t_i(y(v''_i, v'_{-i}))$ for each $v''_i \in V_i$. [15] This implies that $v_i(y(v_i, v'_{-i})) - t_i(y(v_i, v'_{-i})) < v_i(y) - x_i^H$. Together with the supposition of y and the definition of x_i^H, this implies that we can take $v'_i \in V_i$ such that $v_i(y(v_i, v'_{-i})) - t_i(y(v_i, v'_{-i})) < v_i(y(v'_i, v'_{-i})) - t_i(y(v'_i, v'_{-i}))$, that is, $v_i(y(v_i, v'_{-i})) - x_i(v_i, v'_{-i}) < v_i(y(v'_i, v'_{-i})) - x_i(v'_i, v'_{-i})$. This contradicts strategy-proofness.

[14] See Nishizaki (2016) for a proof of this lemma.

[15] Note that we cannot take such a valuation function of the public good by the supposition of y and the strict increasingness of valuation functions of the public good if $x_i^H - t_i(y(v''_i, v'_{-i})) = 0$ for each $v''_i \in V_i$. By Lemma 2, we find that $x_i^H - t_i(y(v''_i, v'_{-i})) < 0$ for each $v''_i \in V_i$ because $y(v''_i, v'_{-i}) = y$ and we have a contradiction to the definition of y if $x_i^H = t_i(y(v''_i, v'_{-i}))$ for some $v''_i \in V_i$.

Situation 2. $y = \sup O_i(v'_{-i})$

Let $x_i^L \equiv \sup\{x_i \in \mathbb{R}_+ | \text{there is } v_i \in V_i \text{ such that } t_i(y(v_i, v'_{-i})) = x_i\}$. By Lemma 2, the definition of x_i^L, and the continuity and strict increasingness of valuation functions of the public good, we can take $v_i \in V_i$ such that $v_i(y) - v_i(y(v''_i, v'_{-i})) > x_i^L - t_i(y(v''_i, v'_{-i}))$ for each $v''_i \in V_i$. [16] This implies that $v_i(y(v_i, v'_{-i})) - t_i(y(v_i, v'_{-i})) < v_i(y) - x_i^L$. Together with the supposition of y and the definition of x_i^L, this implies that we can take $v'_i \in V_i$ such that $v_i(y(v_i, v'_{-i})) - t_i(y(v_i, v'_{-i})) < v_i(y(v'_i, v'_{-i})) - t_i(y(v'_i, v'_{-i}))$, that is, $v_i(y(v_i, v'_{-i})) - x_i(v_i, v'_{-i}) < v_i(y(v'_i, v'_{-i})) - x_i(v'_i, v'_{-i})$. This contradicts strategy-proofness.

Situation 3. Otherwise

Let $x_i^H \equiv \inf\{x_i \in \mathbb{R}_+ | \text{there is } v_i \in V_i \text{ such that } t_i(y(v_i, v'_{-i})) = x_i \text{ and } y(v_i, v'_{-i}) > y\}$ and $x_i^L \equiv \sup\{x_i \in \mathbb{R}_+ | \text{there is } v_i \in V_i \text{ such that } t_i(y(v_i, v'_{-i})) = x_i \text{ and } y(v_i, v'_{-i}) < y\}$. By the supposition of y, we have the following three cases according to whether x_i^H and x_i^L are induced by some valuation function of the public good or not: (i) there is $v_i^L \in V_i$ such that $t_i(y(v_i^L, v'_{-i})) = x_i^L$, but not $v_i^H \in V_i$ such that $t_i(y(v_i^H, v'_{-i})) = x_i^H$, (ii) there is $v_i^H \in V_i$ such that $t_i(y(v_i^H, v'_{-i})) = x_i^H$, but not $v_i^L \in V_i$ such that $t_i(y(v_i^L, v'_{-i})) = x_i^L$, and (iii) there are no $v_i^L, v_i^H \in V_i$ such that $t_i(y(v_i^L, v'_{-i})) = x_i^L$ and $t_i(y(v_i^H, v'_{-i})) = x_i^H$. In the case (i), we know that $y \neq y(v_i^L, v'_{-i})$. Together with Lemma 2, the definition of x_i^H, and the continuity and strict increasingness of valuation functions of the public good, this implies that we can take $v_i \in V_i$ such that $v_i(y(v_i, v'_{-i})) - t_i(y(v_i, v'_{-i})) < v_i(y) - x_i^H$ and have a contradiction by arguments similar to Situations 1 and 2. Similarly, we have a contradiction in the cases (ii) and (iii). □

Lemma 4 shows that the two valuation functions of the public good, whose "peaks" on the option set are equal, induce the same consumption of the public good if the cost sharing scheme satisfies strategy-proofness and Lemma 5 is a well-known result on strategy-proofness and non-bossiness. [17]

Lemma 4. *Suppose that the cost sharing scheme f satisfies **strategy-proofness**. For each $v, v' \in V$ and each $i \in I$, if $v'_i(y(v''_i, v'_{-i})) - t_i(y(v''_i, v'_{-i})) < v'_i(y(v_i, v'_{-i})) - t_i(y(v_i, v'_{-i}))$ for each $y(v''_i, v'_{-i}) \in O_i(v'_{-i}) \setminus \{y(v_i, v'_{-i})\}$, then $y(v_i, v'_{-i}) = y(v'_i, v'_{-i})$.*

Lemma 5. *If the social choice function satisfies **strategy-proofness** and **non-bossiness**, then it is a **cost sharing scheme**.*

A2. Results on Strong Non-Bossiness

Lemma 6 shows the uniqueness of the agent's utility maximizer in the agent's option set under a social choice function satisfying strategy-proofness and strong non-bossiness. [18]

Lemma 6. *Suppose that the social choice function f satisfies **strategy-proofness** and **strong non-bossiness**. For each $v, v' \in V$ and each $i \in I$, if $y(v_i, v'_{-i}) \neq y(v'_i, v'_{-i})$, then $v_i(y(v_i, v'_{-i})) - x_i(v_i, v'_{-i}) > v_i(y(v'_i, v'_{-i})) - x_i(v'_i, v'_{-i})$.*

[16] Note that we can take such a valuation function of the public good even if $x_i^L - t_i(y(v''_i, v'_{-i})) = 0$ for each $v''_i \in V_i$ because $0 < v_i(y) - v_i(y(v''_i, v'_{-i}))$ for each $v''_i \in V_i$ by the supposition of y and the strict increasingness of valuation functions of the public good.

[17] See Nishizaki (2016) for a proof of these lemmas.

[18] See Nishizaki (2016) for a proof of this lemma.

By Proposition 1, we know that strategy-proofness implies the closedness of the option sets under a cost sharing scheme. By imposing strong non-bossiness in addition to strategy-proofness, we also have Proposition 2 showing the convexity.

Proposition 2. *Suppose that the cost sharing scheme f satisfies **strategy-proofness** and **strong non-bossiness**. For each $i \in I$ and each $v'_{-i} \in V_{-i}$, $O_i(v'_{-i})$ is **convex**.*

Proof. Let $i \in I$ and $v'_{-i} \in V_{-i}$. It is sufficient to demonstrate that $\lambda y + (1-\lambda)y' \in O_i(v'_{-i})$ for each $y, y' \in O_i(v'_{-i})$, where $y \neq y'$, and each $\lambda \in (0,1)$. To the contrary, we suppose that there are $y, y' \in O_i(v'_{-i})$, where $y \neq y'$, and $\lambda \in (0,1)$ such that $\lambda y + (1-\lambda)y' \notin O_i(v'_{-i})$. Together with Proposition 1, this implies that we can take $v_i^{\mathrm{L}}, v_i^{\mathrm{H}} \in V_i$ such that

$$[y(v_i^{\mathrm{L}}, v'_{-i}), y(v_i^{\mathrm{H}}, v'_{-i})] \cap O_i(v'_{-i}) = \{y(v_i^{\mathrm{L}}, v'_{-i}), y(v_i^{\mathrm{H}}, v'_{-i})\}, \tag{3}$$

$$y(v_i^{\mathrm{L}}, v'_{-i}) < \lambda y + (1-\lambda)y' < y(v_i^{\mathrm{H}}, v'_{-i}). \tag{4}$$

By (3) and Lemma 2, we can take $v_i^* \in V_i$ such that (iii-2a) $v_i^*(y(v_i^{\mathrm{L}}, v'_{-i})) - v_i^*(y(v''_i, v'_{-i})) \geq t_i(y(v_i^{\mathrm{L}}, v'_{-i})) - t_i(y(v''_i, v'_{-i}))$ for each $v''_i \in V_i$, where $y(v''_i, v'_{-i}) \leq y(v_i^{\mathrm{L}}, v'_{-i})$, and (iii-2b) $v_i^*(y(v_i^{\mathrm{H}}, v'_{-i})) - v_i^*(y(v''_i, v'_{-i})) \geq t_i(y(v_i^{\mathrm{H}}, v'_{-i})) - t_i(y(v''_i, v'_{-i}))$ for each $v''_i \in V_i$, where $y(v''_i, v'_{-i}) \geq y(v_i^{\mathrm{H}}, v'_{-i})$. Together with strategy-proofness, these imply that $v_i^*(y(v_i^{\mathrm{L}}, v'_{-i})) - t_i(y(v_i^{\mathrm{L}}, v'_{-i})) = v_i^*(y(v_i^*, v'_{-i})) - t_i(y(v_i^*, v'_{-i})) = v_i^*(y(v_i^{\mathrm{H}}, v'_{-i})) - t_i(y(v_i^{\mathrm{H}}, v'_{-i}))$, that is, $v_i^*(y(v_i^{\mathrm{L}}, v'_{-i})) - x_i(v_i^{\mathrm{L}}, v'_{-i}) = v_i^*(y(v_i^*, v'_{-i})) - x_i(v_i^*, v'_{-i}) = v_i^*(y(v_i^{\mathrm{H}}, v'_{-i})) - x_i(v_i^{\mathrm{H}}, v'_{-i})$. Together with Lemma 6, this implies that $y(v_i^{\mathrm{L}}, v'_{-i}) = y(v_i^*, v'_{-i}) = y(v_i^{\mathrm{H}}, v'_{-i})$ and contradicts (4). □

By Lemma 3, we know that strategy-proofness implies the lower semi-continuity of the cost sharing scheme. By imposing strong non-bossiness in addition to strategy-proofness, we have Lemma 7 showing the continuity.

Lemma 7. *If the cost sharing scheme f satisfies **strategy-proofness** and **strong non-bossiness**, then it is **continuous**.*

Proof. To the contrary, we suppose that f is not continuous. This implies that there are $i \in I$ and $v'_{-i} \in V_{-i}$ such that t_i is not continuous on $O_i(v'_{-i})$. In addition, there is $v_i^L \in V_i$ such that t_i is not continuous at $y(v_i^L, v'_{-i})$. Together with Lemma 3, this implies that t_i is not upper semi-continuous at $y(v_i^L, v'_{-i})$ and there is $\varepsilon \in \mathbb{R}_+$ such that for each neighborhood $U \subseteq O_i(v'_{-i})$ of $y(v_i^L, v'_{-i})$, $t_i(y') > t_i(y(v_i^L, v'_{-i})) + \varepsilon$ for some $y' \in U$. Because $U \subseteq O_i(v'_{-i})$, this implies that $y(v_i^L, v'_{-i}) < y'$ by Lemma 2. On the basis of the above argument, let $y^H \in (y(v_i^L, v'_{-i}), y')$ be such that we can take $v_i \in V_i$ which satisfies the following condition: $v_i(y(v_i^L, v'_{-i})) - v_i(\lambda y(v_i^L, v'_{-i}) + (1-\lambda)y^H) > t_i(y(v_i^L, v'_{-i})) - t_i(\lambda y(v_i^L, v'_{-i}) + (1-\lambda)y^H)$ for each $\lambda \in (0,1)$. [19] Because $(y(v_i^L, v'_{-i}), y') \subseteq O_i(v'_{-i})$, there is $v_i^H \in V_i$ such that $y(v_i^H, v'_{-i}) = y^H$ and we find that

$$y(v_i^L, v'_{-i}) < y(v_i^H, v'_{-i}) \tag{5}$$

by the definition of y^H. On the basis of the definition of y^H and the continuity and strict increasingness of valuation functions of the public good, we can take $v_i^* \in V_i$ such that (a) $v_i^*(y(v_i^L, v'_{-i})) - v_i^*(y(v''_i, v'_{-i})) \geq t_i(y(v_i^L, v'_{-i})) - t_i(y(v''_i, v'_{-i}))$ for each $v''_i \in V_i$, where $y(v''_i, v'_{-i}) \leq y(v_i^L, v'_{-i})$, (b) $v_i^*(y(v_i^H, v'_{-i})) - v_i^*(y(v''_i, v'_{-i})) \geq$

[19] Note that we can take such a valuation function of the public good by letting y^H be sufficiently close to $y(v_i^L, v'_{-i})$. This requirement is introduced to respect the strict concavity of valuation functions of the public good.

$t_i(y(v_i^H, v'_{-i})) - t_i(y(v''_i, v'_{-i}))$ for each $v''_i \in V_i$, where $y(v''_i, v'_{-i}) \geq y(v_i^H, v'_{-i})$, and (c) $v_i^*(y(v_i^L, v'_{-i})) - v_i^*(\lambda y(v_i^L, v'_{-i}) + (1-\lambda)y(v_i^H, v'_{-i})) > t_i(y(v_i^L, v'_{-i})) - t_i(\lambda y(v_i^L, v'_{-i}) + (1-\lambda)y(v_i^H, v'_{-i}))$ for each $\lambda \in (0,1)$. Together with strategy-proofness, these imply that $v_i^*(y(v_i^L, v'_{-i})) - t_i(y(v_i^L, v'_{-i})) = v_i^*(y(v_i^*, v'_{-i})) - t_i(y(v_i^*, v'_{-i})) = v_i^*(y(v_i^H, v'_{-i})) - t_i(y(v_i^H, v'_{-i}))$, that is, $v_i^*(y(v_i^L, v'_{-i})) - x_i(v_i^L, v'_{-i}) = v_i^*(y(v_i^*, v'_{-i})) - x_i(v_i^*, v'_{-i}) = v_i^*(y(v_i^H, v'_{-i})) - x_i(v_i^H, v'_{-i})$. Together with Lemma 6, this implies that $y(v_i^L, v'_{-i}) = y(v_i^*, v'_{-i}) = y(v_i^H, v'_{-i})$ and contradicts (5). □

A3. Results on Outcome Rectangular Property

On the basis of Lemma 8 showing that the outcome rectangular property can be considered within the range of consumption of the public good under a social choice function satisfying strategy-proofness and non-bossiness, we have Proposition 3 showing the closedness of the range of consumption of the public good under the cost sharing scheme by imposing non-bossiness and the outcome rectangular property in addition to strategy-proofness. [20]

Lemma 8. *Suppose that the social choice function f satisfies **strategy-proofness**, **non-bossiness**, and the **outcome rectangular property**. For each $v, v' \in V$, if $y(v_i, v'_{-i}) = y(v'_i, v'_{-i})$ for each $i \in I$, then $y(v) = y(v')$.*

Proposition 3. *If the cost sharing scheme f satisfies **strategy-proofness**, **non-bossiness**, and the **outcome rectangular property**, then $y(V)$ is **closed**.*

Proof. To the contrary, we suppose that $y(V)$ is not closed. This implies that we can take $y \in \bar{y}(V) \setminus y(V)$, where $\bar{y}(V)$ is the closure of $y(V)$. We have the following three situations according to the relationship between y and $y(V)$.

Situation 1. $y = \inf y(V)$

By Proposition 1, we can take $v \in V$ such that $y(v_i, v_{-i}) = \min O_i(v_{-i})$ for each $i \in I$. [21] In addition, we can take $v' \in V$ such that

$$y < y(v') < y(v) \tag{6}$$

by the supposition of y. For each $i \in I$, we have the following two cases according to the position of $y(v')$ in $O_i(v'_{-i})$ by Proposition 1: (i) $y(v'_i, v'_{-i}) = \max O_i(v'_{-i})$ and (ii) $y(v'_i, v'_{-i}) < \max O_i(v'_{-i})$. In addition, we consider the following two subcases of the case (ii) according to the relationship between $y(v_i, v_{-i})$ and $y(v'_i, v_{-i})$ on the basis of Lemma 4 and the definition of $y(v_i, v_{-i})$: (ii-1) $y(v_i, v_{-i}) = y(v'_i, v_{-i})$ and (ii-2) $y(v_i, v_{-i}) < y(v'_i, v_{-i})$. Let $I_{(\text{i})} \subseteq I$ be the set of agents belonging to the case (i), $I_{(\text{ii-1})} \subseteq I$ be the set of agents belonging to the subcase (ii-1), and $I_{(\text{ii-2})} \subseteq I$ be the set of agents belonging to the subcase (ii-2).

For each $i \in I_{(\text{i})}$, we can take $v_i^* \in V_i$ such that $y(v_i, v_{-i}) = y(v_i^*, v_{-i})$ and $y(v'_i, v'_{-i}) = y(v_i^*, v'_{-i})$ by Lemma 4 and an argument similar to Proposition 2 because $y(v_i, v_{-i}) = \min O_i(v_{-i})$ (see the left hand side of Figure 2). For each $i \in I_{(\text{ii-1})}$, we know that $y(v_i, v_{-i}) = y(v'_i, v_{-i})$ by definition. For each $i \in I_{(\text{ii-2})}$, we can take $v_i^{**} \in V_i$ such that $y(v_i, v_{-i}) = y(v_i^{**}, v_{-i})$ and $y(v'_i, v'_{-i}) = y(v_i^{**}, v'_{-i})$ by Lemma

[20] See Nishizaki (2016) for a proof of this lemma.

[21] Note that we can take such a profile of valuation functions of the public good by letting the slope of v_i on $y(V)$ be sufficiently low for each $i \in I$.

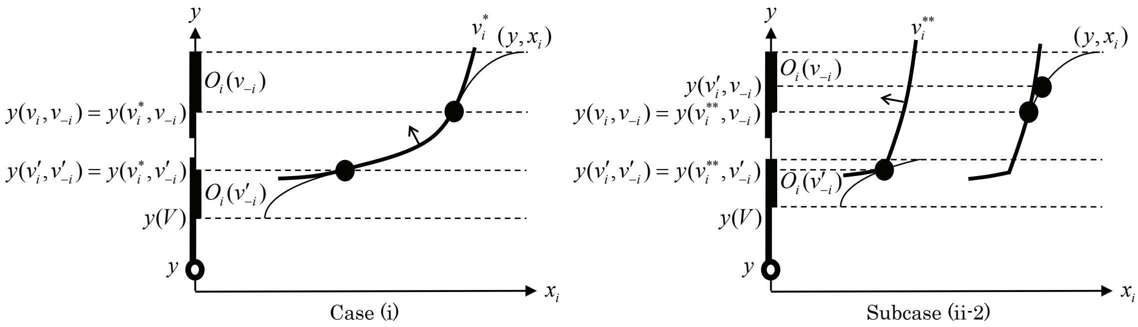

Figure 2: Proof of Situation 1 in Proposition 3

4 and an argument similar to Proposition 2 because $y(v_i, v_{-i}) = \min O_i(v_{-i})$ (see the right hand side of Figure 2). Let $v'' \equiv (v''_{I_{(i)}}, v''_{I_{(ii-1)}}, v''_{I_{(ii-2)}})$ be such that $(v''_{I_{(i)}}, v''_{I_{(ii-1)}}, v''_{I_{(ii-2)}}) = (v^*_{I_{(i)}}, v'_{I_{(ii-1)}}, v^{**}_{I_{(ii-2)}})$. These imply that $y(v_i, v_{-i}) = y(v''_i, v_{-i})$ and $y(v'_i, v'_{-i}) = y(v''_i, v'_{-i})$ for each $i \in I$. Together with Lemma 8, this implies that $y(v) = y(v'') = y(v')$ and contradicts (6).

Situation 2. $y = \sup y(V)$

By Proposition 1, we can take $v \in V$ such that $y(v_i, v_{-i}) = \max O_i(v_{-i})$ for each $i \in I$. In addition, we can take $v' \in V$ such that $y(v) < y(v') < y$ by the supposition of y. For each $i \in I$, we have the following two cases according to the position of $y(v')$ in $O_i(v'_{-i})$ by Proposition 1: (i) $y(v'_i, v'_{-i}) = \min O_i(v'_{-i})$ and (ii) $y(v'_i, v'_{-i}) > \min O_i(v'_{-i})$. In addition, we consider the following two subcases of the case (ii) according to the relationship between $y(v_i, v_{-i})$ and $y(v'_i, v_{-i})$ on the basis of Lemma 4 and the definition of $y(v_i, v_{-i})$: (ii-1) $y(v_i, v_{-i}) = y(v'_i, v_{-i})$ and (ii-2) $y(v_i, v_{-i}) > y(v'_i, v_{-i})$. By an argument similar to Situation 1, we have a contradiction.

Situation 3. Otherwise

Let $U \subseteq Y$ be a neighborhood of y such that $U \cap y(V)$ is convex. This implies that there are the following two cases according the relationship between y and consumption of the public good in $U \cap y(V)$: (i) $y < y''$ for each $y'' \in U \cap y(V)$ and (ii) $y > y''$ for each $y'' \in U \cap y(V)$.

In the case (i), we can take $v \in V$ such that $y(v) \in U$ and $y(v_i, v_{-i}) = \min O_i(v_{-i})$ for each $i \in I$ by Proposition 1. In addition, we can take $v' \in V$ such that $y(v') \in U$ and $y < y(v') < y(v)$ by the supposition of y. By an argument similar to Situation 1, we have a contradiction.

In the case (ii), we can take $v \in V$ such that $y(v) \in U$ and $y(v_i, v_{-i}) = \max O_i(v_{-i})$ for each $i \in I$ by Proposition 1. In addition, we can take $v' \in V$ such that $y(v') \in U$ and $y(v) < y(v') < y$ by the supposition of y. By an argument similar to Situation 2, we have a contradiction. □

Remark 1. In non-excludable public good economies with classical preferences, Serizawa (1996, Lemma) showed the closedness of the range of consumption of the public good under a social choice function satisfying strategy-proofness, non-bossiness, individually rationality, budget-balancedness, and non-exploitation. In other directions, Serizawa (1999, Fact 1) showed it by strategy-proofness, symmetry, and budget-balancedness.

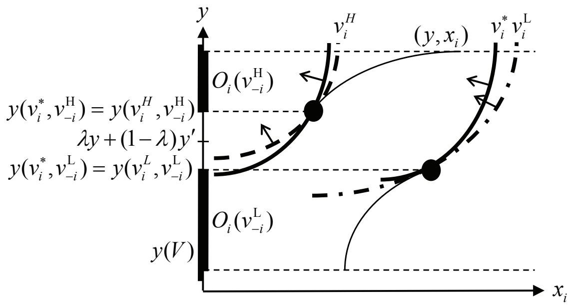

Figure 3: Proof of Proposition 4

By imposing the outcome rectangular property in addition to strategy-proofness and strong non-bossiness, we have Proposition 4 showing the convexity of the range of consumption of the public good under the cost sharing scheme.

Proposition 4. *If the cost sharing scheme f satisfies **strategy-proofness**, **strong non-bossiness**, and the **outcome rectangular property**, then $y(V)$ is **convex**.*

Proof. Let $y, y' \in y(V)$ and $\lambda \in [0,1]$. We have the following three cases according to the value of λ: (i) $\lambda = 0$; (ii) $\lambda = 1$; and (iii) $\lambda \in (0,1)$. In the case (i), we know that $\lambda y + (1-\lambda)y' = y' \in y(V)$. In the case (ii), we know that $\lambda y + (1-\lambda)y' = y \in y(V)$. In the case (iii), we have the following two subcases according to the relationship between y and y': (iii-1) $y = y'$ and (iii-2) $y \neq y'$. In the subcase (iii-1), we know that $\lambda y + (1-\lambda)y' \in y(V)$.

The remainder of this proof demonstrates that $\lambda y + (1-\lambda)y' \in y(V)$ in the subcase (iii-2). To the contrary, we suppose that $\lambda y + (1-\lambda)y' \notin y(V)$. On the basis of Proposition 3, we can take $v^{\mathrm{L}}, v^{\mathrm{H}} \in V$ such that for each $i \in I$,

$$[y(v^{\mathrm{L}}), y(v^{\mathrm{H}})] \cap O_i(v_{-i}^{\mathrm{L}}) = \{y(v^{\mathrm{L}})\}, \tag{7}$$

$$[y(v^{\mathrm{L}}), y(v^{\mathrm{H}})] \cap O_i(v_{-i}^{\mathrm{H}}) = \{y(v^{\mathrm{H}})\}, \tag{8}$$

$$y(v^{\mathrm{L}}) < \lambda y + (1-\lambda)y' < y(v^{\mathrm{H}}). \tag{9}$$

By (7), (8), and Lemma 2, we can take $v^* \in V$ such that for each $i \in I$, (iii-2a) $v_i^*(y(v_i^{\mathrm{L}}, v_{-i}^{\mathrm{L}})) - v_i^*(y(v_i'', v_{-i}^{\mathrm{L}})) > t_i(y(v_i^{\mathrm{L}}, v_{-i}^{\mathrm{L}})) - t_i(y(v_i'', v_{-i}^{\mathrm{L}}))$ for each $y(v_i'', v_{-i}^{\mathrm{L}}) \in O_i(v_{-i}^{\mathrm{L}}) \setminus \{y(v_i^{\mathrm{L}}, v_{-i}^{\mathrm{L}})\}$ and (iii-2b) $v_i^*(y(v_i^{\mathrm{H}}, v_{-i}^{\mathrm{H}})) - v_i^*(y(v_i'', v_{-i}^{\mathrm{H}})) > t_i(y(v_i^{\mathrm{H}}, v_{-i}^{\mathrm{H}})) - t_i(y(v_i'', v_{-i}^{\mathrm{H}}))$ for each $y(v_i'', v_{-i}^{\mathrm{H}}) \in O_i(v_{-i}^{\mathrm{H}}) \setminus \{y(v_i^{\mathrm{H}}, v_{-i}^{\mathrm{H}})\}$ (see Figure 3). Together with Lemma 4, this implies that $y(v_i^*, v_{-i}^{\mathrm{L}}) = y(v_i^{\mathrm{L}}, v_{-i}^{\mathrm{L}})$ and $y(v_i^*, v_{-i}^{\mathrm{H}}) = y(v_i^{\mathrm{H}}, v_{-i}^{\mathrm{H}})$ for each $i \in I$. Together with Lemma 8, this implies that $y(v^{\mathrm{L}}) = y(v^*) = y(v^{\mathrm{H}})$ and contradicts (9). □

Remark 2. The combination of Lemmas 1 and 7 and Proposition 4 implies the continuity of a cost sharing scheme satisfying strategy-proofness, strong non-bossiness, and the outcome rectangular property on the range of consumption of the public good.

A4. Proof of Theorem 3

Let f be a social choice function satisfying strategy-proofness, strong non-bossiness, and the outcome rectangular property. By Lemma 5, we know that f is a cost sharing scheme. By Propositions 3 and

4, we know that the range of consumption of the public good is closed and convex. These imply that the problem of providing a divisible and non-excludable public good with the cost shares is reduced to a voting environment in which the set of alternatives is equivalent to the range of consumption of the public good, which is a closed interval. In addition, we know the continuity of f on the range of consumption of the public good, as stated in Remark 2. This implies that each utility function induces a continuous preference defined over the range of consumption of the public good. Together with the result of Barberà and Peleg (1990, Theorem 3.1), these imply that f is dictatorial if the range of consumption of the public good contains at least three alternatives. If not, then f is constant because the range of consumption of the public good is closed and convex.

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